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THREE-WAVE INTERACTIONS OF DISTURBANCES IN A SUPERSONIC BOUNDARY LAYER

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Within the framework of the weakly nonlinear stability theory, group interaction of disturbances in a supersonic boundary layer is considered. The disturbances are represented by two spatial packets of traveling instability waves (wave trains) with multiple frequencies. The possibility of energy redistribution in such wave systems in the case of three-wave resonant interactions of packet constituents is considered. The model is used to test the dynamics of unstable waves arising due to introduction of controlled high-intensity disturbances into a supersonic boundary layer. It is found that this mechanism is not the main one for the features of streamwise dynamics of such nonlinear waves being observed.

Key words: *supersonic boundary layer, nonlinear disturbances, three-wave resonant interactions.*

Introduction. Mechanisms of processes inherent in the laminar–turbulent transition in supersonic boundary layers are extensively studied. This became possible after experimental investigations of nonlinear stages of disturbance evolution and construction of theoretical models on the basis of experimental data.

Important results were obtained in studying the nonlinear evolution, the process of spectrum filling, and identification of carrier frequencies for finite disturbances that cannot be considered as linear. For a supersonic boundary layer, subharmonic instability is observed if the level of controlled disturbances is rather low [1, 2]. The laws of this subharmonic instability are qualitatively and quantitatively described by the nonlinear model of interaction in resonant triads [3–6]. Three-dimensional modes prevail in the spectrum, and its filling is a cascade process of identification of three-dimensional subharmonics in the parametric region.

If controlled disturbances of rather high intensity are introduced into a supersonic boundary layer on a flat plate at a Mach number M $= 2$ [7], however, a situation arises that differs considerably from that described above. The downstream evolution of such disturbances was called “anomalous” [7]. In the experiments of [7], it was found that the initial spectrum of disturbances contains two wave packets with multiple frequencies (subharmonic frequency $f_1 = 10$ kHz and fundamental frequency $f_2 = 20$ kHz); the packet with the frequency f_1 dominates. The wave packets are wave trains [8] with a wide spectrum in terms of the transverse wavenumber β, which contain three-dimensional waves propagating at angles $-90^\circ \leqslant \chi \leqslant 90^\circ$ to the main flow direction [$\chi = \arctan(\beta/\alpha^r)$, where α^r is the longitudinal wavenumber]; the plane component with $\beta = 0$ has the highest intensity. These wave trains are rather symmetric in terms of β within the entire interval under study. Further downstream, the two-dimensional character of the wave spectra remains unchanged, though the linear stability theory predicts that the growth rates of three-dimensional waves are much higher than the growth rates of plane waves. An increase in intensity is observed for both three-dimensional and two dimensional waves, being particularly significant for the latter. Kosinov et al. [7] believe that the experimental results described are caused by nonlinearity of the streamwise dynamics of high-intensity oscillations.

In the present work, an attempt is made to find out whether it is possible to explain the above-described phenomena by nonlinear interaction of the own traveling Tollmien–Schlichting disturbances within the framework of interactions in resonant triads on the basis of the weakly nonlinear stability theory used to explain the dynamics of disturbances at early stages of nonlinearity.

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Basic Formulas and Methods of Solution. The initial postulates of the nonlinear model of interaction in resonant triads for compressible boundary layers are described in detail in [3, 4]. Following [3, 4], we consider disturbed fields of velocity u, v, w, density ρ_0, pressure p_0, and temperature T_0 of a compressible gas

$$u = U(Y) + \varepsilon u', \qquad v = \varepsilon v', \qquad w = \varepsilon w', \qquad \rho_0 = \rho(Y) + \varepsilon \varrho',$$
$$p_0 = P + \varepsilon p', \qquad T_0 = T(Y) + \varepsilon \Theta', \qquad p'/P = \varrho'/\rho + \Theta'/T \tag{1}$$

in a dimensionless coordinate system [the thickness $\delta = \sqrt{\mu_e x/(U_e \rho_e)}$ is the characteristic linear size and μ is the dynamic viscosity; the subscript e indicates free-stream parameters; the primed and nonprimed quantities are the fluctuating and mean components of the corresponding quantities; the scale parameter is $\varepsilon \ll 1$]. Normalization is performed to flow parameters at the outer boundary; the Reynolds and Mach numbers based on these parameters are $\mathrm{Re} = \sqrt{x \rho_e U_e/\mu_e}$ and $\mathrm{M} = U_e/a_e$ (a is the velocity of sound). The dimensionless values of the longitudinal coordinate X coincide with the value of Re. The method for obtaining distributions of U and T in the laminar flow is described in [9]; $\rho = 1/T$.

The solution is constructed by the method of expansion in terms of the small parameter ε and two-scale expansion of the x coordinate. In addition to the "fast" scale X, we introduce the "slow" scale $\xi = \varepsilon X$ characterizing the difference in the rates of variation of the disturbance phase and amplitude. The possibility of introducing the "slow" scale is conditioned by the large difference in the velocities mentioned.

We seek the solution for waves of the type

$$u' = A(\xi) u(Y) \exp(i\theta), \qquad \theta = \int \alpha \, dX + \int \beta \, dZ - \omega t, \tag{2}$$

where u' is the streamwise component of velocity, A is the amplitude slowly changing along the streamwise coordinate, $u(Y)$ is the amplitude eigenfunction, $\alpha = \alpha^r + i\alpha^i$ ($\alpha^i < 0$ is the growth rate), and the frequency $\omega = 2\pi f$ is a real quantity.

Substituting Eqs. (1) and (2) into the full system of equations of motion and conservation for a compressible gas [9], within the framework of the weakly nonlinear theory, we obtain the initial system for disturbances:

$$[\rho(Gu + U_Y v) + i\alpha p/(\gamma \mathrm{M}^2) - (\mu/\mathrm{Re}) u_{YY}] \exp(i\theta) = F_u,$$
$$[\rho G w + i\beta p/(\gamma \mathrm{M}^2) - (\mu/\mathrm{Re}) w_{YY}] \exp(i\theta) = F_w,$$
$$[\rho G v + p_Y/(\gamma \mathrm{M}^2)] \exp(i\theta) = F_v, \qquad [G\varrho + \rho_Y v + \rho(i\alpha u + v_Y + i\beta w)] \exp(i\theta) = F_p, \tag{3}$$
$$[\rho(G\Theta + T_Y v) + (\gamma - 1)(i\alpha u + v_Y + i\beta w) - \mu\gamma/(\sigma \mathrm{Re}) \Theta_{YY}] \exp(i\theta) = F_\Theta,$$
$$\varrho = \rho(p/P - \Theta/T), \qquad G = i(-\omega + \alpha U).$$

Here $\gamma = C_P/C_V$ is the ratio of specific heats, $\sigma = C_P \mu_e/K$ is the Prandtl number, and K is the thermal conductivity. For $F = 0$, Eqs. (3) form a linearized system for three-dimensional disturbances [9].

The nonlinear terms in (3) have the following form:

$$F_u = \rho(u' u'_X + v' u'_Y + w' u'_Z) + \varrho'(u'_t + U u'_X + U_Y v'),$$
$$F_v = \rho(u' v'_X + v' v'_Y + w' v'_Z) + \varrho'(v'_t + U v'_X),$$
$$F_w = \rho(u' w'_X + v' w'_Y + w' w'_Z) + \varrho'(w'_t + U w'_X),$$
$$F_p = \varrho'(u'_X + v'_Y + w'_Z) + u' \varrho'_X + v' \varrho'_Y + w' \varrho'_Z,$$
$$F_\Theta = \varrho'(\Theta'_t + U\Theta'_X + T_Y v') + \rho(u'\Theta'_X + v'\Theta'_Y + w'\Theta'_Z) + 2\gamma(\gamma - 1)\mathrm{M}^2 p'(u'_X + v'_Y + w'_Z).$$

The nonlinear effects determine the terms quadratic in amplitude in the nonlinear terms.

The boundary conditions for disturbances are

$$u = v = \Theta = 0, \qquad Y = 0, \qquad Y = \infty.$$

In the first-order approximation in terms of ε, the homogeneous system (3) is the basic one for finding the eigenvalues of α for given values of the frequency ω and Reynolds number Re and also for constructing the amplitude functions of the linear waves of the form (2) with normalization $|v|_{kn} = 1$, where v_{kn} is the value of v in the final coordinate of integration $Y_{kn} = 15$ (the boundary-layer thickness corresponds to $Y = 7$). In the weakly nonlinear theory, these parameters of the linear waves are considered to be sought, and nonlinearity affects only the amplitude A.

First, let us analyze the nonlinear evolution of disturbances satisfying the conditions of phase synchronism, for which $\theta_j = \theta_l + \theta_k$. Usually, $j \neq l \neq k$ but there may be triplets with $l = k$. For such three-wave systems, the amplitude equations can be obtained using a standard procedure of averaging and solvability conditions [4]:

$$\frac{dA_j}{d\xi} = -\alpha_j^i A_j + \sum_{l,k}^{N} S_{l,k} A_l A_k \exp\left(-i\Delta_{j,l,k}\right),$$
$$\frac{dA_l}{d\xi} = -\alpha_l^i A_l + \sum_{j,k}^{N} S_{j,k} A_j A_k^* \exp\left(i\Delta_{j,l,k}\right), \qquad \frac{dA_k}{d\xi} = -\alpha_k^i A_k + \sum_{j,l}^{N} S_{j,l} A_j A_l^* \exp\left(i\Delta_{j,l,k}\right). \tag{4}$$

The coefficients S expressed through the nonlinear terms F of system (3) characterize the force field generated by interacting waves. The terms with Δ take into account possible detuning of the wave relation in triplets in terms of α^r:

$$\Delta_{j,l,k} = \mathrm{Real}\Big(\int (\alpha_j - \alpha_l - \alpha_k)\, dX\Big).$$

We write the complex amplitudes A in a trigonometric form

$$A = a\exp(i\psi), \qquad a = |A|, \qquad \psi = \arg A$$

and solve Eqs. (4) with respect to a and ψ. The initial conditions for the amplitudes are set in accordance with the experimental distributions of mass velocities of the waves (intensities I) in the initial cross section X_0. The relation between the oscillation amplitudes a and intensities I can be expressed via the calculated values of fluctuations of mass velocity of the waves $m = \rho u + \varrho U$:

$$I_j(X_0) = a_j(X_0) m_j(Y_{\max}) \exp(-\alpha_j^i X_0).$$

The value $Y = Y_{\max}$ corresponds to the transverse coordinate with the maximum value of m of the most intense two-dimensional wave with the subharmonic frequency. For the values of Re considered, this coordinate remained constant: $Y_{\max} = 4.35$. The initial phases were assumed to be arbitrary; in the basic variant, $\psi_j(X_0) = 0$.

Results and Discussion. In the experiments of [7], the measurements were performed in the range $x = 60$–110 mm, which corresponded to the values of the Reynolds number $624 \leqslant \mathrm{Re} \leqslant 846$. The stagnation temperature was constant and equal to 310 K, $\gamma = 1.4$, and $\sigma = 0.72$. The dimensionless frequencies were introduced by means of the frequency parameter F ($\omega = F\,\mathrm{Re}$), thus, we had $F_1 = 19.2 \cdot 10^{-6}$ and $F_2 = 2F_1$. The calculations were performed for the same parameters.

We considered the group interaction of wave trains obtained in [7]. Integration in (2) was replaced by summation; therefore, the real wave trains in terms of transverse wavenumbers β were replaced by a set of N discrete wave modes. In the present work, we had $N = 7$. For each wave mode, the dimensionless wavenumbers $b = 10^{-3}\beta/\mathrm{Re}$ were constant. Figure 1 shows the scheme of division of the wave trains ($b = 0$ corresponded to the most energy-carrying two-dimensional components).

Figure 2 shows the growth rates $-\alpha^i$ of linear waves with the frequency parameters F_1 for the subharmonic and F_2 for the fundamental frequency within the considered range of Reynolds numbers for the examined values of b. The initial position of the two-dimensional component of the subharmonic is the region near the lower branch of the neutral curve; its linear growth rate increases with increasing Re and does not reach a maximum in the final measurement cross section. The growth rates of three-dimensional components are usually much higher than the growth rates of this mode. The highest growth rates are observed for the three-dimensional mode with $b_1 = 0.077$.

For the initial value of Re, the growth rate of the main wave with the parameter F_2 (Fig. 2b) is close to the maximum of the linear growth rate in the region of instability; further downstream, its values approach the upper branch of the neutral curve. As in Fig. 2a, the growth rates of three-dimensional modes are higher than the growth rates of the plane component. This feature is violated only for the three-dimensional mode with a large azimuthal

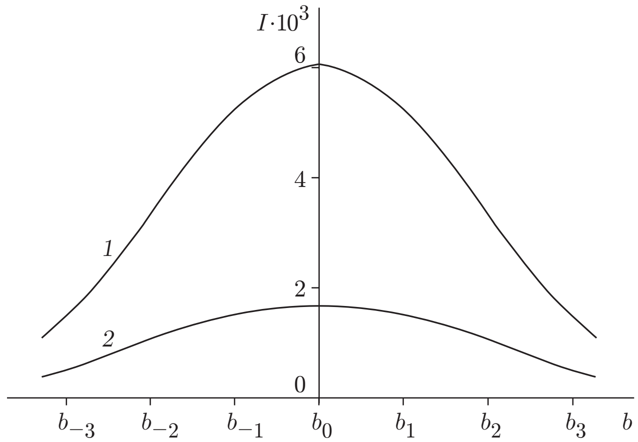

Fig. 1. Scheme of division of the wave trains into discrete components of the considered azimuthal wave parameters b for the dimensionless frequency parameters F_1 (1) and F_2 (2).

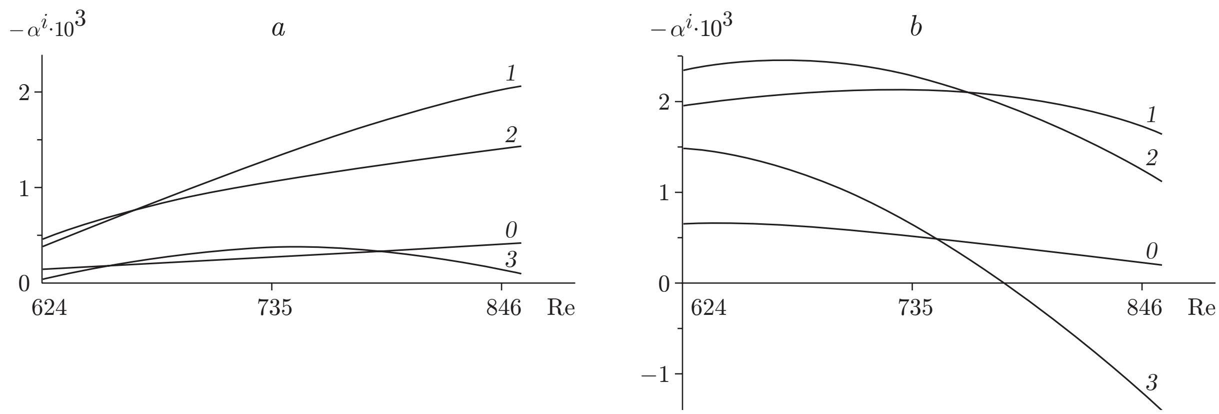

Fig. 2. Linear growth rates $-\alpha^i(\mathrm{Re})$ of the considered modes with the frequency parameters F_1 (a) and F_2 (b): curve 0 is the two-dimensional component ($b_0 = 0$) and curves 1–3 are the three-dimensional components [$b_1 = 0.077$ (1), $b_2 = 0.154$ (2), and $b_3 = 0.231$ (3)].

parameter $b_3 = 0.231$: oblique waves inclined at an angle $\chi \approx 85°$ to the plane of the flow; the intensity of these waves is not very high.

For seven components of the wave trains, the conditions of phase synchronism allow us to form 20 triplets (the superscripts 1 and 2 indicate phases of modes with the frequency parameters F_1 and F_2, respectively; the subscripts correspond to the subscripts of the parameter b):

$$\theta_0^2 = \theta_0^1 + \theta_0^1, \qquad \theta_0^2 = \theta_1^1 + \theta_{-1}^1, \qquad \theta_0^2 = \theta_2^1 + \theta_{-2}^1, \qquad \theta_0^2 = \theta_3^1 + \theta_{-3}^1, \qquad \theta_1^2 = \theta_0^1 + \theta_1^1, \qquad \theta_1^2 = \theta_2^1 + \theta_{-1}^1,$$

$$\theta_1^2 = \theta_3^1 + \theta_{-2}^1, \qquad \theta_2^2 = \theta_0^1 + \theta_2^1, \qquad \theta_2^2 = \theta_3^1 + \theta_{-1}^1, \qquad \theta_2^2 = \theta_1^1 + \theta_1^1,$$

$$\theta_3^2 = \theta_0^1 + \theta_3^1, \qquad \theta_3^2 = \theta_2^1 + \theta_1^1, \qquad \theta_{-1}^2 = \theta_0^1 + \theta_{-1}^1, \qquad \theta_{-1}^2 = \theta_1^1 + \theta_{-2}^1, \qquad \theta_{-1}^2 = \theta_2^1 + \theta_{-3}^1, \tag{5}$$

$$\theta_{-2}^2 = \theta_0^1 + \theta_{-2}^1, \qquad \theta_{-2}^2 = \theta_1^1 + \theta_{-3}^1, \qquad \theta_{-2}^2 = \theta_{-1}^1 + \theta_{-1}^1,$$

$$\theta_{-3}^2 = \theta_0^1 + \theta_{-3}^1, \qquad \theta_{-3}^2 = \theta_{-2}^1 + \theta_{-1}^1.$$

With allowance for the symmetry of the wave trains, we can confine ourselves to considering the triplets for positive values of b only; the number of triplets decreases to 12 thereby.

Figure 3 shows the wave detuning $\Delta(\mathrm{Re})$ for typical triplets. As compared to the values of α^r, this detuning is not large; therefore, these triplets can affect the values of the corresponding amplitudes determined by Eqs. (4).

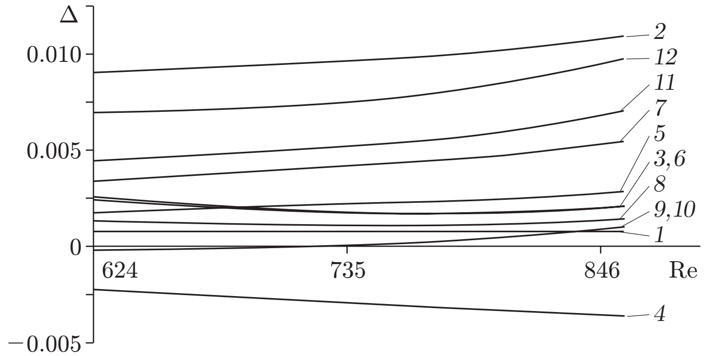

Fig. 3. Detuning in terms of wavenumbers in triplets (5).

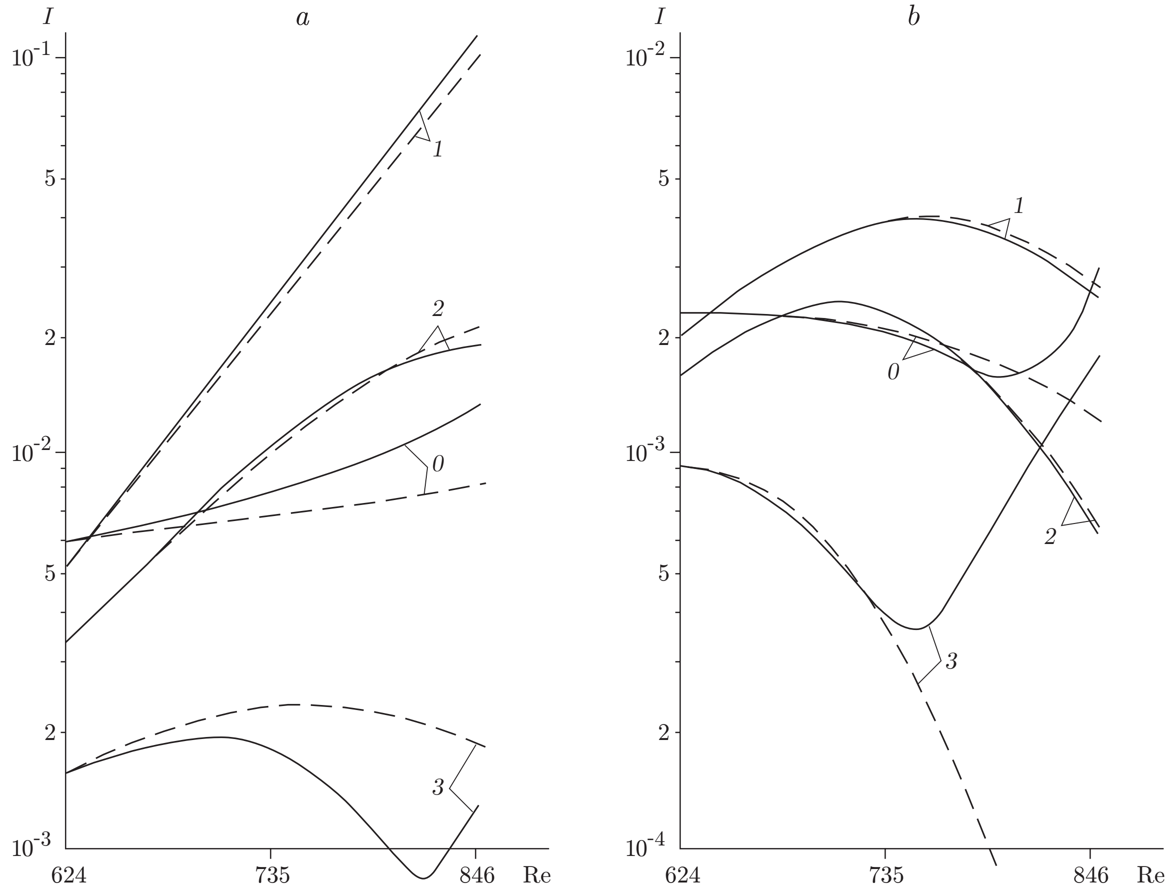

Fig. 4. Dynamics of intensities of the considered modes with the frequency parameters F_1 (a) and F_2 (b): curve 0 is the two-dimensional component ($b_0 = 0$) and curves 1–3 are the three-dimensional components [$b_1 = 0.077$ (1), $b_2 = 0.154$ (2), and $b_3 = 0.231$ (3)]; the solid and dashed curves refer to the nonlinear and linear models, respectively.

Within the framework of the model considered, we analyze the resultant nonlinear intensities I of the wave modes with the parameter F_1 (Fig. 4a). The dashed curves correspond to calculations by the linear model: $I_{lin}(X) = a(X_0)m(Y_{\max}) \exp(-\alpha^i X)$.

Allowance for nonlinearity leads to an increase in intensity of the plane wave with the subharmonic frequency; for Re = 846, the value of I is 1.5 times greater than that calculated by the linear model. For the most intense three-dimensional component with $b_1 = 0.077$, the calculation results of intensity by the nonlinear model show that its value exceeds that calculated by the linear model by a factor of 1.2. At the same time, the mode with $b_2 = 0.154$ develops almost linearly. It follows from Fig. 4a that the intensity of the three-dimensional mode with $b_3 = 0.231$ decreases on the major part of the interval considered, and its values are significantly lower than those calculated by the linear model.

Thus, in the case of group interaction, energy redistribution is directly proportional to the angles of inclination of the wave modes to the plane of the main flow; the greatest portion of energy is obtained by the plane wave. This process can lead to wave-train constriction in terms of the transverse wavenumbers at the subharmonic frequency, which corresponds to the process actually observed. At the same time, the portion of energy obtained by the plane wave is not large, and the nonlinear interaction in resonant triplets cannot be the reason for the predominant growth of this component observed in reality.

The nonlinear dynamic of components of the main wave with the parameter F_2 has a smaller effect on the process of wave interaction because of the considerably lower initial intensities (Fig. 4b). In this case, a significant increase in I is observed for the two-dimensional mode for $\mathrm{Re} > 800$; therefore, for $\mathrm{Re} = 846$, the intensity of the two-dimensional component becomes higher than the intensity of the three-dimensional mode, which agrees with the data of [7]. It follows from Fig. 4b that three-dimensional modes with $b_1 = 0.077$ and $b_2 = 0.154$ develop almost linearly in accordance with the dependences $\alpha^i(\mathrm{Re})$ (see Fig. 2b).

It should be noted that some features obtained in analyzing interactions of wave trains in the regime of three-wave resonant systems is in qualitative agreement with the observed streamwise dynamics of wave packets. Nevertheless, the nonlinear interaction considered is not the main one in the process of energy redistribution in high-intensity wave trains. Obviously, the physical processes inherent in the downstream evolution of such disturbances are more complicated. An adequate theoretical description of these processes requires unification of the resonant and combinatorial second-order models of interaction [10] and also investigation of the influence of disturbances of different types, in particular, steady disturbances [11] whose existence was noted in [7], on the wave-packet dynamics.

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